UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE

STEP MATHEMATICS PAPER 2: 9470: JULY 2005

REPORT FOR CENTRES

General Remarks

Almost all levels of achievement in this examination were represented significantly. Thus as much as a third of all candidates failed to complete more than two questions, whereas about a fifth completed at least four questions. In particular, the best candidates showed considerable mathematical potential.

Again, it is good to be able to record that the practice of contravening the rubric by handing in responses to more than six questions is still in decline and this trend continues to benefit candidates generally. In this respect, it was good to see that some candidates made quality their first priority and so achieved an excellent overall script total from fewer than six questions.

The distribution of question choice was extremely non-uniform, as was the case last year, and the proportion of candidates who concentrate exclusively on Section A appears to be on the increase. In any case, it is clear that the best work usually comes from Section A and the worst from Section C.

The standard of presentation of work continues to decline to the extent that some material is unreadable. The idea that what is marked is necessarilly restricted to what can be read does not appear to be universally understood. Most of the candidature has probably had little experience of an examination of this type and so found it difficult to set out well structured solutions. In passing therefore, it seems reasonable to suggest that future candidates should acquire mathematical communication skills as a necessary preliminary.

An inevitable consequence of chaotic working was the proliferation of elementary errors in consequence of which some candidates wandered into intractable mathematical situations. Here checking and perhaps a fresh start is needed, not a mindless drive forward into a dead end.

Three particular skills were substandard. They were, the approximation of a function $f(x)$ for small x by a polynomial, the correct working of inequalities (these featured in all sections) and the presentation of a clear, well annotated force diagram in a mechanical situation.

Nevertheleless, despite the above criticisms, one can say that much impressive work was in evidence. Many candidates showed courage and determination in the face of an onslaught of difficult mathematics.

Comments on responses to individual questions

SECTION A: PURE MATHEMATICS

Q1 This was the most popular question of the paper and the majority of candidates made some progress with it. Many responses were undermined by elementary errors.

At the outset, the preliminary result $d[x^2e^{-x^2}]/dx = 2xe^{-x^2} - 2x^3e^{-x^2}$ appeared in almost all responses. Remarkably, however, not all candidates were able to identify all the roots of $x - x^3 = 0$.

A small minority of candidates thought that $P(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ whereas the correct starting point is $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2)$ $(*)$. Actually, some wrote $x(x^2 - a)(x^2 - b)$ for $x(x^2 - a^2)(x^2 - b^2)$.

Beyond this beginning, the best candidates could see immediately that $P(x)$ can take the form $-x^4/2 + px^2 + q$ and this approach certainly simplified later working. In contrast, some started with $P(x) \equiv \sum_{i=0}^{4} c_i x^i$, but this more complicated strategy generated many erroneous solutions.

There were also those who attempted to obtain $P(x)$ from $e^{x^2} \int x(x^2 - a^2)(x^2 - b^2)e^{-x^2}\, dx$. The systematic and correct use of the integration by parts rule will readily show how for $I_n = \int x^{2n+1} e^{-x^2}\, dx$, I_1 and I_2 relate to $I_0 = -(1/2)e^{-x^2}$ in a simple way. In this context, few responses were at all clear and generally notational confusions led to inaccuracy.

$x = -1, 0, 1 :$ $P(x) =$ any non-zero scaling of $-x^4/2+(a^2/2+b^2/2-1)x^2+-1+a^2/2+b^2/2-a^2b^2/2$

Q2 This question was also popular and most responses made significant progress with the majority of the sections available.

(a) (i) The correct results for $f(12)$ and $f(180)$ appeared in almost all responses.

(ii) The working here was often hazy and protracted. Notational confusions led to poor reasoning. What was required was an argument based on

$$N = p_1^{\alpha_1} \dots p_k^{\alpha_k} \Rightarrow \dots \Rightarrow f(N) = p_1^{\alpha_1 - 1} \dots p_k^{\alpha_k - 1}(p_1 - 1)\dots(p_k - 1).0$$

(b) In all three parts of this section, it was expected that, at least, the conclusion would be made clear. However, there were many instances where this did not happen.

(i) Most responses showed a suitable counterexample, e.g., $f(3)f(9) = 2 \times 6 = 12 \neq f(27) = 18$, and thus proved that the displayed result lacks generality.

(ii) There were few failures here. The correct conclusion was usually supported by a simple argument such as $f(p)f(q) = p(1 - 1/p)q(1 - 1/q) = pq(1 - 1/p)(1 - 1/q) = f(pq)$.

(iii) Most responses groped there way through working to show, e.g., $f(5) = 4$, $f(6) = 2$, $f(30) = 2 \times 4 = 8$, from which the required conclusion can be made. However, some candidates were unable to do this. In this respect there were erroneous statements of the form $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ and especially, $(P' \Rightarrow Q') \Rightarrow (P \Rightarrow Q)$.

(c) Responses generally started with $p^{m-1}(p - 1) = 146410$, but some led on to incorrect results, or simply faded out. A popular incorrect conclusion was $p = 11$, $m = 4$.

(a) (i) $f(12) = 4$, $f(180) = 48$: (b) (i) not always true, (ii) true, (iii) false: (c) $p = 11$, $m = 5$.

Q3 This very popular question provided an opportunity for candidates to show their competence with basic A-level mathematics. Only the last part was at all unusual. Unexpectedly, most responses showed at least one error in (ii), but contained sound mathematics to establish the final inequality.

For the introductory inequality it is only necessary to state that $y(0) = 0$, $y(\pi/2) = 1$ and to show that $dy/dx = 0$ at the origin and that $dy/dx > 0$ for $0 < x \leq \pi/2$. The majority of responses proceeded in this way, though the layout of the working was unclear in some instances. The majority of sketch graphs were incorrect and/or incomplete. Common errors were a non-zero gradient at the origin and/or a zero gradient at the point $(\pi/2, 1)$.

(i) Most responses showed accurate and complete working to establish the displayed result.

(ii) The working in almost all responses soon led to the preliminary result that $J = \int_0^{\pi/2} y^2\, dx = I_1 - I_2 + I_3$, where $I_1 = \int_0^{\pi/2} \sin^2 x\, dx$, $I_2 = \int_0^{\pi/2} x \sin 2x\, dx$, $I_3 = \int_0^{\pi/2} x^2 \cos^2 x\, dx$.

The results $I_1 = \pi/4$, $I_2 = \pi/4$ were usually in evidence, though sometimes these followed from erroneous working. In contrast, many candidates were unable to supply the extensive technical detail needed for the determination of I_3. Thus a complete and correct evaluation of J appeared in only a minority of responses.

Q4 Though less popular than its predecessors it nevertheless generated some good work. The very best solutions sailed through the first part, used the displayed result there to establish the second displayed result and then produced a carefully worked solution for the final part. However, such quality mathematics came only from a minority.

In contrast the majority did not understand how the second part relates to the first and so started all over again. Moreover, it was common for an unchecked solution to appear. Even if this was correct, it could not be awarded full credit for the simple reason that in this context there are several plausible possibilities.

Q5 This was not a popular question and some candidates made absolutely no progress.

Some responses at least got as far as establishing that $2r = b + c - a$ and so went on to get a result for R in terms of a, b, c. The transposition of the expression obtained into a quadratic function, Q, of q proved to be beyond most.

The calculus based responses to the final part were usually incomplete in that the nature of the stationary point of Q was not considered. Those who used an exclusively algebraic method fared better. Generally, this strategy was worked with impressive accuracy and the interpretation of the result obtained for Q in terms of the displayed inequality was usually satisfactory.

Q6 This too was an unpopular question and very few responses were complete and correct. It appeared that some candidates were unfamiliar with the concept of a general term and yet others did not work easily with the summation operator Σ.

(i) Those who were familiar with the basic terminology appertaining to infinite series got through the first sentence of the question though, in some cases, with a lot of unnecessary labour.

Most responses showed an attempt to uses some, or all, of $S_j = (1-x)^{-j}$, $j = 1, 2, 3$ in order to evaluate $\sum_{n=1}^{\infty} n2^{-n}$ and $\sum_{n=1}^{\infty} n^2 2^{-n}$.

(ii) Almost all candidates who got this far produced sufficient working to show that the general term of $(1-x)^{-1/2}$ can be put into the form displayed in the question.

They usually went on to put $x = 1/3$ in order to evaluate the first of the 2 series in the final part of the question and generally worked accurately. However, the evaluation of the second series proved to be much more difficult. The majority of responses at least hinted at a correct overall strategy, namely differentiation followed by setting $x = 1/3$, but lack of technical expertise undermined much of the working. Nevertheless, it must be said, this evident lack of mathematical competence was not a feature of the responses to *Q6* alone.

(i) x^n, $(n+1)x^n$, $(1/2)(n+1)(n+2)x^n$: 2, 6.

(ii) $\sqrt{3/2}$, $(1/4)\sqrt{3/2}$.

Q7 This was the least well answered question of Section A and this was due mainly to a lack of understanding of the geometry supplemented by defective expertise in dealing with inequalities involving trigonometric functions.

(i) The description of the locus of P was usually correct as far as it went. Basically, what was required was the identification of it as a circle, the radius of this circle, the location of the centre and the plane in which it is contained. The same is required for the locus of Q, but here, some candidates thought that it is an ellipse and were usually unable to specify the plane in any geometrically intelligible way.

(ii) Most responses showed the correct formation of a relevant scalar product in terms of t. Thereafter there was some illegal working of the trigonometry based on the incorrect resolution of $\cos(.)\cos(..)$ into a sum of cosines and $\sin(.)\sin(..)$ into a difference of cosines. Moreover much energy was expended on finding OQ even when the candidate had already concluded in (i) that it is constant and equal to 3, and even more remarkably, the proving that $OQ = 3$ did not lead to any correction of the statement in (i) that Q describes an ellipse.

(iii) Generally, failure in (ii) did not inhibit sensible attempts at this concluding section of the question. Nevertheless, few candidates seemed to realise that 1 cycle in the $t-$ domain corresponds to 2 cycles in the $\theta-$ domain. Certainly candidates would have helped themselves considerably if they had worked from an accurate sketch graph of the cosine function. As it was, very few did this and for the most part fell back on hazy inequality arguments which, more often than not, were inconsistent with basic properties such as $\cos\theta$ is decreasing over the open interval $(0, \pi)$ and is increasing over the open interval $(\pi, 2\pi)$.

(i) P describes a circle centre O and radius 1 in the $x - y$ plane.

Q describes circle centre O and radius 3 in the plane $\sqrt{3}x - z = 0$.

Q8 This question led to application of essentially correct methodologies. However, at the technical level there many inaccuracies.

Most responses soon arrived at something like $A - 1/y = \int x^3(1 + x^2)^{-5/2}\,dx$.

At the integration stage, at least six methods were in evidence. The most popular of these was based on the integration by parts rule. This obvious and simple strategy was generally applied accurately. Also in evidence were the use of substitutions such as $u = x^2$, $v = 1 + x^2$, $x = \tan t$, $x = \sinh w$ and a consideration of the derivative of $1/y = (2 + 3x^2)/[3(1 + x^2)^{3/2}]$.

This last method would have been completely acceptable had the solution not been displayed in the question. In this context, however, it must necessarily be regarded as verification and, as such, did not merit full credit.

Application of the given initial condition was usually accurate if only because the displayed result was at hand.

The large positive x approximation was established in a rigorous way by only a minority. Following the expansion of $(1 + x^2)^{-3/2}$, terms were prematurely discarded and then brought back into the working again in some illegal way so as to produce the displayed approximation.

The sketch of C was often deficient in some way in that the gradient at (0,1) was non-zero and the location of the horizontal asymptote was not identified.

A substantial minority of candidates failed to observe that the second differential equation can be obtained immediately from the first by the substitution $y = z^2$. Such a deduction makes the drawing of the second diagram, involving 2 branches, a simple matter.

SECTION B: MECHANICS

Q9 Few responses showed a clear and complete diagram for both parts of this question and no doubt this deficiency was the main cause of bad modelling. Candidates generally needed to think through the application of Newton's laws of motion in this context in order to ensure that all terms were present and this is what some failed to do.

(i) A particular direction of P, usually parallel to the slope, was assumed in about half of all responses. Such a presupposition trivialised the question and so very little credit could be given for this strategy.

In contrast, the best responses went very rapidly to the fundamental equation

$$P\cos\theta = mg + mg/2 + (1/2\sqrt{3})(mg\sqrt{3}/2) + (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta),$$

where $\theta + \pi/6$ is the angle which P makes with the horizontal. Here, calculus methods were often used in order to determine the optimal direction of P, in the sense of the question, and very often such arguments were incomplete and almost incoherent.

The superior method of first writing

$$P\sin(\theta + \pi/3) = (11\sqrt{3}/8)mg$$

followed by the use of $\sin(.) \leq 1$ was not to be seen to any great extent and this would suggest that lack of facility in the working of trigonometric expressions was a partial cause of failure in this question.

(ii) Much the same comments carry over to the situation here where the critical equation is

$$P\cos\theta = mg + mg/2 - (1/2\sqrt{3})(mg\sqrt{3}/2) - (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta).$$

However, few responses at this stage were complete and correct.

(i) Direction for least magnitude of P is parallel to the slope.

(ii) Least magnitude of P is $(\sqrt{3}/8)mg$.

Q10 This turned out to be the most popular and the best answered question of Section B.

The key to the obtaining of a completely correct response is the setting up of displacement-time equations which take into account both the different time origins and the different displacement origins of the two missiles. Without a correct form of these equations, or some equivalent, it is impossible to make significant progress with this question. With these, accurate working will speedily take the candidate to a complete and correct solution. As it was, most responses made almost no progress, though on the other hand, a few were complete in every respect and well presented.

The four basic equations which describe the horizontal and vertical components of the motion of the particles are essentially:

$$x_1 = 80t,\ y_1 = 60t - 5t^2, x_2 = 180 - 120(t - T),\ y_2 = 160(t - T) - 5(t - T)^2 \ \ (t \geq T)\ \ (*)$$

In this context, some candidates used $t + T$ and t in place of t and $T - t$, respectively, but then left themselves with problems with regard to the interpretation of results at the end of their analysis.

Almost all candidates who got as far as (*), or some equivalent, worked from $x_1 = x_2$ and $y_1 = y_2$ to establish (eventually) a quadratic equation in T alone. It was in this process that many inaccuracies, such as one would not expect at this level, occurred. Beyond that, a small minority went on to obtain the roots $T = 1,\ 90,$ but hardly anyone could produce an effective argument as to why, in fact, $T = 1$.

Q11 This question generated mainly incomplete responses. As with *Q9*, there was a dearth of useful and properly annotated diagrams. Streamlined versions of Newton's second law of motion appeared in some solutions.

For the obtaining of the first result there usually appeared a correct equation of motion for each particle. For the particle on the slope this included an accurate specification of the frictional force opposing the motion. After some algebra the first result appeared. Candidates would have found it helpful, both here and later in the question, to have denoted the pervasive constant $(m_2 - m_1)/(m_2 + m_1)$ by a single letter, say λ.

The particle on the slope begins the second phase of the motion with speed $u = \lambda g T$ and so the total time to the highest point is $(1 + \lambda)T$.

For the final part of the question, the information given leads to the key equation

$$(g/10)(1+\lambda)^2T^2 = (\lambda g/2)T^2 + (\lambda^2 g/2)T^2. \ (*)$$

Provided g is set to 10 (ms^{-2}), as directed by the question, it is easy to solve (*) for λ and hence for m_1/m_2.

$m_1/m_2 = 3/5$

SECTION C: PROBABILITY AND STATISTICS

Q12 This was the most popular question of Section C. Attempts generally lacked coherence and this was true whether responses were exclusively symbolic and/or a tree diagram was employed. Very few candidates produced complete and correct working for all sections.

In the first part, most responses produced a valid argument to establish the meaning of the basic quantity $G = ap + bq$ and from there could establish the first result.

Both (ii) and (iii) can be worked easily in terms of G, or by use of a tree diagram, but progress was usually negated by lack of a clear meaningful notation and also by the suposition that it was the same twin who responded to both e-mails.

Q13 Few candidates made significant progress with this question. Responses generally began with an explanation as to why $p = e^{-\lambda}(1+\lambda)$, but generally did not include any serious attempt to show that Y is a binomial variate.

(i) It was expected that candidates would get as far as

$$q \approx 1 - (1+\lambda)[1 - \lambda + \lambda^2/2)]$$

when λ is small. However, as in *Q8*, candidates seemed to be unfamiliar with the concept of approximating a function by use of its power series expansion and so there were few satisfactory solutions here.

(ii) Similarly, few candidates got beyond

$$P(Y = n) = p^n > 1 - \lambda \Rightarrow e^{-n\lambda}(1 + \lambda)^n > 1 - \lambda$$

to write something like

$$-n\lambda + n\lambda - n\lambda^2/2 > -\lambda - \lambda^2/2 + O(\lambda^3)$$

and so go on to establish the required result.

(iii) Some responses began (correctly) with

$$P(Y > 1|Y > 0) = P(Y > 1)/P(Y > 0) = (1 - q^n - npq^{n-1})/(1 - q^n)$$

but again, most candidates lacked the technical expertise needed for further progress.

Q14 This was the least popular question of the paper. Less than half of all responses got as far as establishing the displayed result for σ^2.

Some approximately sensible sketch graphs appeared for (i), but the number of those candidates who made further progress was almost zero.